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An LMI Formulation of Robustness Analysis for Systems with Time-Varying and LTI Uncertainty

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Abstract

A procedure is developed to analyze uncertain systems having both time-varying (TV) and linear, time-invariant (LTI) uncertainty. The problem is formulated as a two step procedure involving the solution of two sets of LMIs using both constant and frequency dependent scales to account for the different types of uncertainty. In the first step, the scales corresponding to the TV parameters are constrained to be constant over a set of frequencies and both constant and frequency-dependent scales are computed to minimize the peak value of the robustness bound for the chosen frequencies. In the second step, the constant scales are held fixed and the frequency-dependent scales corresponding to the LTI uncertainty are computed to minimize the robustness bound at each frequency. Numerical examples are given to demonstrate the procedure, which gives a less conservative result than previously shown and thus allows for more accurate analysis of systems with TV and LTI uncertainty.

1 Introduction

Control law design for systems operating over a wide range of conditions is traditionally a heuristic process consisting of several steps. First, control laws are synthesized for a representative set of operating conditions by fixing the system's parameters in each design. The resulting set of control laws is then implemented in the parameter-varying system by scheduling with operating condition. This gain scheduling can consist of either switching between fixed control laws or transitioning gradually from one to another using interpolation. Stability of the closed-loop system is achieved by sufficient robustness margins at each design point and can be evaluated through exhaustive simulation, although there is no guarantee of stability for all operating conditions and time-consuming redesign is often necessary. Furthermore, as the complexity of the system increases or as more sophisticated control law design techniques are used, these steps can become difficult, as the resulting control laws can become more difficult to gain schedule because of their increased or even varying dynamic order.

To overcome these obstacles, Packard [7] developed a technique to address the problem of gain scheduling controllers by representing the plant as a linear parameter-varying (LPV) system. Here, the parameter-varying system is represented as the linear fractional transformation of a nominal system with the time-varying parameters. The controller is then parameterized using the same set of time-varying parameters so that the resulting controller is scheduled *a priori* with operating condition and hence the problem of gain scheduling a set of LTI controllers can be eliminated. The TV parameters in the plant and the controller parameterization are combined and the problem is reformulated into the paradigm of a nominal plant with linear

fractional uncertainty. This formulation allows small gain synthesis techniques such as $\mathcal{H}_\infty$ or μ-synthesis to be used. However, when this LPV structure is combined with the traditional LTI uncertainties such as actuator uncertainty, sensor uncertainty, unmodeled dynamics, or a fictitious performance block the resulting system has both TV and LTI uncertainty and the traditional $\mathcal{H}_\infty$ or μ-synthesis framework cannot account for both simultaneously. Hence, there is a need for a nonconservative method of analyzing such systems. While this framework was the main motivation behind developing a less conservative analysis technique, there are other types of systems with both TV and LTI uncertainty.

The remainder of this paper is organized in four sections. Necessary background theory is discussed briefly in Section 2. In Section 3 we develop a procedure to compute a robustness bound for systems with both TV and LTI uncertainty that is less conservative than currently used methods. While not presented here, the analysis procedure developed in Section 3 can be used with a synthesis procedure similar to DKD iteration presented in [9] to synthesize controllers that are less conservative and provide better performance than techniques currently being used. This analysis procedure is demonstrated on some numerical examples in Section 4. These examples show that the new method gives a better robustness bound than other techniques. In Section 5 we conclude with a brief summary and a discussion of some lessons learned.

2 Robustness Analysis

In this section we recall some basic results in robustness theory. Consider a stable transfer function $G(s)$ in a feedback interconnection with an uncertain matrix $\Delta \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ has some block structure and $\bar{\sigma}(\Delta) < \gamma^{-1}$ for all $\Delta \in \mathbf{\Delta}$. The primary objective of robustness analysis is to determine the largest Δ for which the sta-

bility of the closed-loop system is guaranteed. From the small gain theorem, we know that the closed-loop system is asymptotically stable if

$$\bar{\sigma}(G(\jmath\omega)) \leq \gamma,$$

for all $\omega \in \mathbb{R}$. Thus, the maximum singular value, $\bar{\sigma}(G(\jmath\omega))$, gives a frequency-dependent robustness bound, the peak of which defines a lower bound on the smallest destabilizing Δ.

The above robustness measure can be quite conservative. Fortunately, the conservatism of this robustness bound can be reduced by choosing invertible scalings that commute with the uncertain matrix Δ. Let $D \in \mathcal{D}$, where $\mathcal{D}$ is the set of invertible scalings that commute with every $\Delta \in \mathbf{\Delta}$. Note that if $D \in \mathcal{D}$ and $\Delta \in \mathbf{\Delta}$, then $D\Delta D^{-1} = DD^{-1}\Delta = \Delta$. In this case, the closed-loop system is stable if there exists $D \in \mathcal{D}$ such that

$$\bar{\sigma}(DG(\jmath\omega)D^{-1}) \leq \gamma$$

for all $\omega \in \mathbb{R}$.

Now define,

$$\gamma(\omega) \triangleq \bar{\sigma}(DG(\jmath\omega)D^{-1}),$$

and

$$\bar{\gamma} \triangleq \sup_\omega \gamma(\omega) = \sup_\omega \bar{\sigma}(DG(j w)D^{-1}),$$

so that $\bar{\gamma}$ is the peak of $\gamma(\omega)$, the scaled maximum singular value. Then, $\bar{\gamma}^{-1}$ is a lower bound on the size of the smallest destabilizing Δ. Specifically, stability is guaranteed for all Δ satisfying $\bar{\sigma}(\Delta) < \bar{\gamma}^{-1}$, that is, there is no Δ satisfying $\bar{\sigma}(\Delta) < \bar{\gamma}^{-1}$ such that $\det(sI - G(\jmath\omega)\Delta)$ is singular for some $\omega \in \mathbb{R}$. Then, the scalings can be used to minimize $\bar{\gamma}$, or equivalently, to maximize the size of Δ for which stability can be guaranteed.

Suppose $\Delta \in \mathbf{\Delta}$ represents LTI uncertainty. In this case, the scaling matrix $D_\Delta \in \mathcal{D}$ can be

chosen to be a function of frequency so that the peak of the robustness bound is given by

$$\bar{\gamma} = \sup_{\omega} \ \bar{\sigma}(D_\Delta(\omega)G(\jmath\omega)D_\Delta^{-1}(\omega)).$$

Hence the robustness analysis problem is to find the frequency-dependant scale $D_\Delta(\omega)$ that minimizes the robustness bound

$$\gamma(\omega) = \bar{\sigma}(D_\Delta(\omega)G(\jmath\omega)D_\Delta^{-1}(\omega))$$

pointwise at each frequency. This can be computed as the solution to a convex optimization problem as in [5].

In contrast, suppose $\theta \in \boldsymbol{\Delta}$ represents TV uncertainty. In this case, the scaling D_θ must be fixed for all frequency. In this case the peak of the robustness bound is given by

$$\bar{\gamma} = \sup_{\omega} \ \bar{\sigma}(D_\theta G(\jmath\omega)D_\theta^{-1}).$$

Hence the analysis problem is to find the constant scale D_θ that minimizes $\bar{\gamma}$, the peak of the robustness bound $\gamma(\omega) = \bar{\sigma}(D_\theta G(jw)D_\theta^{-1})$ over frequency. This problem can be formulated as a convex problem in the state space as in [3]. Alternatively, if the plant has the LPV form in [7] the analysis can be done as a step of the controller synthesis using the method presented in [2].

In many cases, the uncertain matrix $\Delta \in \boldsymbol{\Delta}$ may represent uncertainty having both LTI and TV blocks. In this case, the augmented uncertainty $\tilde{\Delta}$ is

$$\tilde{\Delta} = \begin{bmatrix} \theta & 0 \\ 0 & \Delta \end{bmatrix}.$$

Invertible scalings having the complementary block structure can be defined as

$$\tilde{D} = \begin{bmatrix} D_\theta & 0 \\ 0 & D_\Delta(\omega) \end{bmatrix}.$$

The analysis problem is then to find the constant and frequency dependent scales D_θ and $D_\Delta(\omega)$

that minimize the peak of the robustness bound given by

$$\bar{\gamma} = \sup_{\omega} \ \bar{\sigma}\left(\tilde{D}G(\jmath\omega)\tilde{D}^{-1}\right). \tag{1}$$

Furthermore, since the frequency dependent scales that minimize $\bar{\gamma}$ are only unique at the frequency at which $\bar{\gamma}$ occurs, additional analysis information can be obtained at other frequencies by finding the frequency-dependent scales $D_\Delta(\omega)$ that minimize

$$\gamma(\omega) = \bar{\sigma}\left(\tilde{D}G(\jmath\omega)\tilde{D}^{-1}\right), \tag{2}$$

at each $\omega \in \mathbb{R}$. The method to obtain the scales for this case is not as straightforward since some of the scales must be constant over all frequencies while others can vary with frequency. The procedures described for the case of only TV uncertainty could be applied here by treating LTI uncertainty as TV; however, this could be conservative. In order to minimize the conservatism, frequency-dependent scales must be used for the LTI blocks.

Some of the conservatism introduced by treating all of the uncertainty as TV and using a method such as in [3] to obtain all constant scales can be reduced by performing a second step which scales the LTI blocks with frequency-dependent scales. This $3Ds$ procedure is summarized below. Using a state space formulation, find the constant scale D_θ and the constant scale $D_{\Delta c}$ that minimize

$$\sup_{\omega} \ \bar{\sigma}\left(\begin{bmatrix} D_\theta & 0 \\ 0 & D_{\Delta c} \end{bmatrix} G(\jmath\omega) \begin{bmatrix} D_\theta^{-1} & 0 \\ 0 & D_{\Delta c}^{-1} \end{bmatrix} \right).$$

Then, at each frequency find the frequency dependent scales, $D_\Delta(\omega)$, minimizing,

$$\gamma(\omega) = \bar{\sigma}\left(\tilde{D}G(\jmath\omega)\tilde{D}^{-1}\right).$$

Finally, the peak value of this robustness bound gives $\bar{\gamma}$. This technique has the advantage that it can be used to analyze any system containing TV and LTI uncertainties.

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Similarly, in [9], the method of [2] was augmented with a second step which scales the LTI blocks with frequency-dependent scales, thus reducing the conservatism. This DKD technique presented in [9] can be summarized as follows. First, let $G(s) = F_L(P(s), K(s))$ where $P(s)$ is an LPV system and $K(s)$ is an LPV controller, both in linear fractional form as described in [7]. Using the technique of [2], find the constant scale, D_θ, and the controller minimizing

$$\sup_\omega \bar{\sigma}\left(\begin{bmatrix} D_\theta & 0 \\ 0 & I \end{bmatrix} G(\jmath\omega) \begin{bmatrix} D_\theta^{-1} & 0 \\ 0 & I \end{bmatrix}\right).$$

Then, at each frequency find the frequency dependent scales, $D_\Delta(\omega)$, minimizing,

$$\gamma(\omega) = \bar{\sigma}\left(\tilde{D} G(\jmath\omega) \tilde{D}^{-1}\right).$$

Finally, $\bar{\gamma}$ is given by the peak value of the robustness bound $\gamma(\omega)$. Note that this procedure is usually used in an iterative manner and works well for synthesis for the class of problems described in [7]. However, the analysis is a byproduct of synthesis, which is a rather limiting constraint as often it is desirable to analyze a controller's robustness to uncertainties that weren't accounted for during synthesis. Additionally, there are systems that contain TV uncertainties for which the method in [2] doesn't apply.

Unfortunately, both of the above methods can introduce significant conservatism, since there is no guarantee that the constant scale found in the first step is the best constant scale once the system is augmented with the frequency-dependent scales obtained in the second step. This is easy to verify by computing the second step using different constant scales. In [1] an alternative approach is given, where the frequency-dependent scale is parameterized as a rational function and the optimization problem is reformulated using results such as those in [4, 8]. This procedure can be very computationally intensive even for problems of moderate size, and the dynamic order of the frequency-dependent

scale must be chosen *a priori*. If too low an order is specified the results could be arbitrarily conservative. On the other hand, the computational time becomes prohibitive for systems of even moderate order. At present, this technique doesn't seem practical and was not used for comparison in our numerical work.

3   Analysis with Varying and Uncertain Parameters

In this section we describe a new procedure for computing the optimal mixed, constant and frequency dependent scales for systems with both TV and LTI uncertainty, that is, we describe a procedure that gives the least conservative results possible for the analysis problem described by equations (1) and (2). The problem is formulated as a two step procedure involving the solution of two sets of LMIs. In the first step, the scales corresponding to the TV parameters are constrained to be constant over a set of frequencies. Then, both constant and frequency-dependent scales that minimize the peak of the scaled maximum singular value are computed for the chosen frequencies. In the second step, the constant scales are held equal to the value found in the first step and the optimal frequency-dependent scales corresponding to the LTI uncertainty are computed to minimize the scaled maximum singular value at each frequency. It should be noted that the peak of the robustness bound cannot be reduced in the second step. Thus, for analysis purposes where only the peak value of the robustness bound is needed, only step one needs to be completed.

Using the definition of the maximum singular value, it follows that there exists an invertible scaling D such that $\bar{\sigma}(DG(\jmath\omega)D^{-1}) \leq \gamma$ if and only if there exists a positive definite scaling Q such that $G^*(\jmath\omega)QG(\jmath\omega) \leq \gamma^2 Q$, where $Q \in \mathcal{D}$ and $D = Q^{1/2}$. Thus, the system is

stable for all $\Delta \in \boldsymbol{\Delta}$ satisfying $\bar{\sigma}(\Delta) < \gamma^{-1}$ if $G^*(\jmath\omega)QG(\jmath\omega) \leq \gamma^2 Q$, where $Q \in \mathcal{D}$.

Now, the analysis problem for systems with both TV and LTI uncertainties described in equations (1) and (2) can be restated. Let the scaling matrix Q be partitioned as

$$Q = \begin{bmatrix} Q_\theta & 0 \\ 0 & Q_\Delta(\omega) \end{bmatrix},$$

where Q_θ is the constant scaling matrix corresponding to the TV parameter and $Q_\Delta(\omega)$ is the frequency-dependent scaling matrix that corresponds to the LTI uncertainty. Then, for mixed TV and LTI uncertainty, the analysis problem is to find Q_θ and $Q_\Delta(\omega)$ in

$$G^*(\jmath\omega)QG(\jmath\omega) \leq \gamma^2(\omega)Q$$

that minimizes the peak of $\gamma(\omega)$ for all $\omega \in \mathbb{R}$ and minimizes $\gamma(\omega)$ at all other frequencies. The two step procedure used to perform this analysis can be summarized as follows. First find Q_θ and $Q_\Delta(\omega)$ minimizing $\bar{\gamma}$ such that

$$G^*(\jmath\omega)QG(\jmath\omega) \leq \bar{\gamma}^2 Q$$

for all $\omega \in \mathbb{R}$. This gives the optimal constant scale, $D_\theta = Q_\theta{}^{1/2}$. Then, at each frequency ω find a new $Q_\Delta(\omega)$ that minimizes $\gamma(\omega)$ at that frequency such that

$$G^*(\jmath\omega)QG(\jmath\omega) \leq \gamma^2(\omega)Q.$$

This gives the optimal frequency-dependent scale, $D_\Delta(\omega) = Q_\Delta{}^{1/2}(\omega)$.

These two steps are refered to as the 'DD' procedure and are implemented using LMIs as follows.

1. Choose a set of frequencies ω_i and evaluate the system frequency response $G(\jmath\omega_i)$ at those frequencies. Form the system of LMIs

$$\begin{aligned} Q_\theta &> 0, \\ Q_\Delta(\omega_i) &> 0, \quad i = 1, \dots, n, \\ G^*(\jmath\omega_i)Q_iG(\jmath\omega_i) &< \bar{\gamma}^2 Q_i, \quad i = 1, \dots, n, \end{aligned}$$

where

$$Q_i = \begin{bmatrix} Q_\theta & 0 \\ 0 & Q_\Delta(\omega_i) \end{bmatrix}$$

is the scaling matrix for the ith frequency. Then, find the scalings Q_i, $i = 1, \dots, n$, that minimize $\bar{\gamma}$ subject to the system of LMIs.

2. For each frequency, form the system of LMIs

$$\begin{aligned} Q_\Delta(\omega_i) &> 0, \\ G^*(\jmath\omega_i)Q_iG(\jmath\omega_i) &< \gamma_i{}^2 Q_i, \end{aligned}$$

using the value of Q_θ found in the previous step. Then, at each frequency ω_i, find the scaling $Q_\Delta(\omega_i)$ that minimizes γ_i subject to the system of LMIs.

The first set of LMIs minimizes the peak of the robustness bound over the set of chosen frequencies to yield the suboptimal value of the constant scaling Q_θ. As the number of frequencies increases, Q_θ approaches its optimal value. The second set of LMIs minimizes the robustness bound at each frequency ω_i to yield the optimal values of the frequency-dependent scales $Q_\Delta(\omega_i)$.

4 Numerical Examples

In this section we show two numerical examples using the procedure presented in the previous section. The first example is used to demonstrate the procedure, while the second example shows that the procedure is less conservative than other methods used to analyze this class of problems.

4.1 Example 1

In this example we use a simple, low order model to demonstrate the analysis procedure described

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in Section 3. The analysis was performed on a weighted, closed-loop LPV model of the VISTA F-16 short period dynamics in linear fractional form with time-varying parameter θ and time-invariant uncertainty Δ_{unc}. The LPV model is valid for an altitude of 1000 feet and a Mach number between $M_{min} = 0.35$ and $M_{max} = 0.65$. The controller minimizes an $\mathcal{H}_\infty$ norm for the linear model with Mach at its nominal value of 0.5. The state space realization of the model is given in the Appendix.

The parameter θ is a 2×2 time-varying block containing a normalized Mach number, δM, on its diagonal, so that

$$\theta = \begin{bmatrix} \delta M & 0 \\ 0 & \delta M \end{bmatrix},$$

where $\delta M(t) \in [-1\ 1]$ corresponds to a Mach number between M_{min} and M_{max}. As the Mach number varies, the short period model changes to represent the short period dynamics at different flight conditions. The uncertainty Δ_{unc} is scalar and is augmented with a full 3×3 fictitious LTI performance block, Δ_{perf}. Thus, the augmented uncertainty $\tilde{\Delta}$ is

$$\tilde{\Delta} = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \Delta_{unc} & 0 \\ 0 & 0 & \Delta_{perf} \end{bmatrix}.$$

The invertible scalings associated with this block structure can be defined as

$$\tilde{D} = \begin{bmatrix} D_\theta & 0 & 0 \\ 0 & d_{\Delta 1}(\omega) & 0 \\ 0 & 0 & d_{\Delta 2}(\omega)I \end{bmatrix},$$

where D_θ is a 2×2 constant matrix and $d_{\Delta 1}(\omega)$ and $d_{\Delta 2}(\omega)$ are frequency-dependent scalars. By letting,

$$\Delta = \begin{bmatrix} \Delta_{unc} & 0 \\ 0 & \Delta_{perf} \end{bmatrix},$$

and

$$D_\Delta(\omega) = \begin{bmatrix} d_{\Delta 1}(\omega) & 0 \\ 0 & d_{\Delta 2}(\omega)I \end{bmatrix},$$

the analysis problem is in the form of equation (1). Then, the DD analysis technique presented in Section 3 can be directly applied to obtain the suboptimal scalings D_θ and $D_\Delta(\omega)$. In the first step of the analysis constant and frequency-dependent scales that minimize the peak of the robustness bound are found simultaneously. This produces the suboptimal constant scale. Then, in the second step, using the suboptimal constant scale, new frequency-dependent scales are found that minimize the robustness bound at each frequency. This yields the optimal frequency-dependent scales for the given suboptimal constant scale. The robustness bound after each of the two steps is shown in Figure 1. Note that the peak of the bound was minimized in the first step but at other frequencies the bound was minimized in the second step. The optimal frequency-dependent scales $d_{\Delta 1}(\omega)$ and $d_{\Delta 2}(\omega)$ obtained after the second step are shown in Figure 2. The constant scale D_θ is

$$D_\theta = \begin{bmatrix} 1.0000 & -0.0027 \\ -0.0027 & 0.2094 \end{bmatrix}.$$
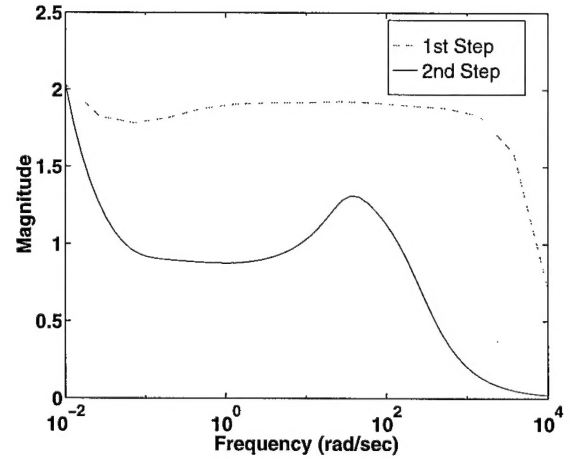

Figure 1: Example 1, DD Robustness Bounds

The robustness bound obtained using the DD procedure can be compared with the limiting cases obtained by treating all uncertainty as TV and using optimal constant scales and by treating all uncertainty as LTI and using optimal frequency dependant scales. These limiting cases
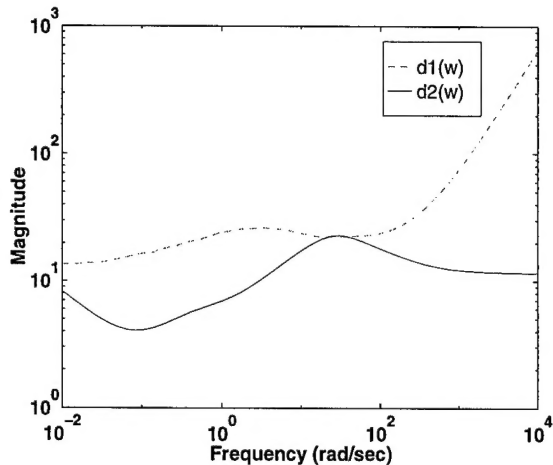
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Figure 2: Example 1, Frequency Dependent Scales

are shown in Figure 3. The bound obtained using the DD analysis technique is between the other two bounds at each frequency as it should be. In this example, the DD bound is essentially equal to the bound obtained using all frequency dependant scales. It can be seen that assuming all TV uncertainty would be conservative.
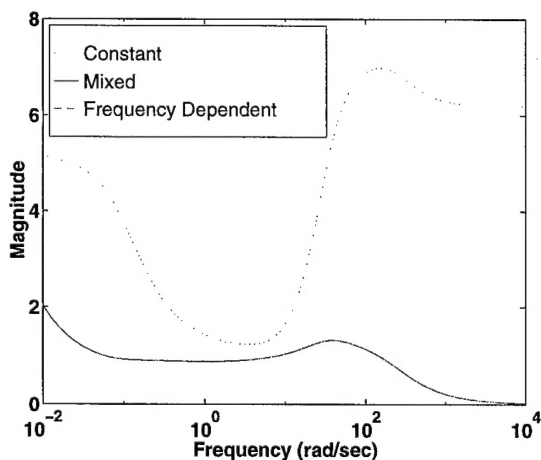

Figure 3: Example 1, Robustness Bounds for Different Classes of Uncertainty

4.2 Example 2

In this example the closed loop system obtained after the fourth DKD iteration in [9] is analyzed

and compared to the results presented there. The robustness bound obtained for this system after the first and second steps described in Section 3 are shown in Figure 4. As in the first example, it can be seen that the peak of the bound is not lowered in the second step. The DD analysis results are compared with the limiting cases as discussed in the previous example and are shown in Figure 5. The analysis of this system, which has both TV and LTI uncertainty, demonstrates that treating all uncertainty as TV can be conservative, while treating it all as LTI can give misleading robustness bounds. Thus, in order to get accurate analysis information, the type of each uncertainty must be properly accounted for by the analysis procedure.
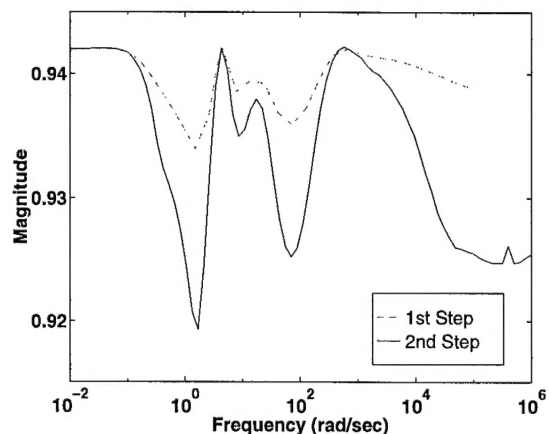

Figure 4: Example 2, DD Robustness Bounds

In [9], the DKD procedure outlined in Section 2 was used for analysis. The robustness bound using this technique is shown in Figure 6 along with the robustness bound obtained using the DD analysis procedure. The peak of the DD bound is less than the peak of the bound in [9] because the constant and frequency-dependent scales that minimize the peak of the robustness bound were found simultaneously, whereas in the DKD procedure, the constant scales are found independently of the frequency-dependent scales. When the DKD procedure is used in an iterative manner as in [9], the constant scales found in
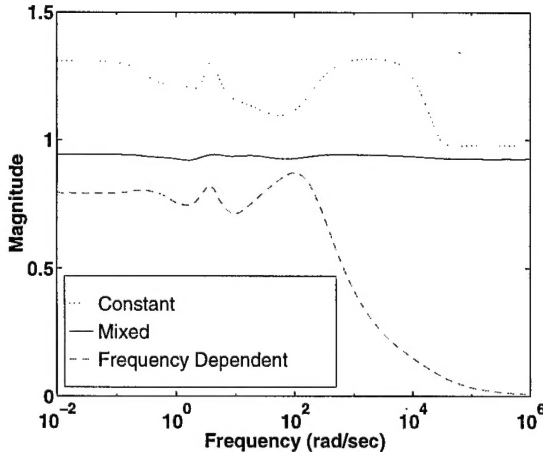
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Figure 5: Example 2, Robustness Bounds for Different Classes of Uncertainty

later iterations are improved due to the frequency-dependent scales which have been absorbed into the system. Evidence of this can be seen in Figure 7, where the constant scales are found without the aid of frequency-dependent scales from a previous iteration. However, even when the DKD procedure is used iteratively until it converges as in [9] the results can be conservative as shown here.
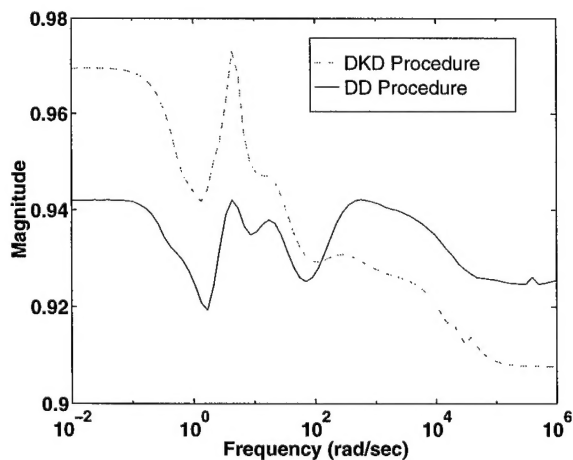

Figure 6: Example 2, Robustness Bounds using DKD and DD

We also compare the DD robustness bound with the bound obtained using the $3Ds$ procedure outlined in Section 2. The $3Ds$ procedure is the only feasible analysis alternative to the DD procedure for general systems containing both TV and LTI uncertainty, that is, for systems for which the controller was not synthesized using the technique of [2] or [9]. The bounds are shown in Figure 7. Again, the peak of the robustness bound obtained by the DD procedure is less than that obtained by an alternative procedure. Here the difference is even more dramatic than shown in Figure 5. This is because as with the DKD method, the $3Ds$ procedure finds the constant scales independent of the frequency dependent ones, but unlike the DKD procedure, the $3Ds$ method doesn't benefit from frequency-dependent scales obtained in a previous iteration. It can be seen that the DD analysis procedure can remove a significant amount of conservatism when analyzing a general system with both TV and LTI uncertainty.
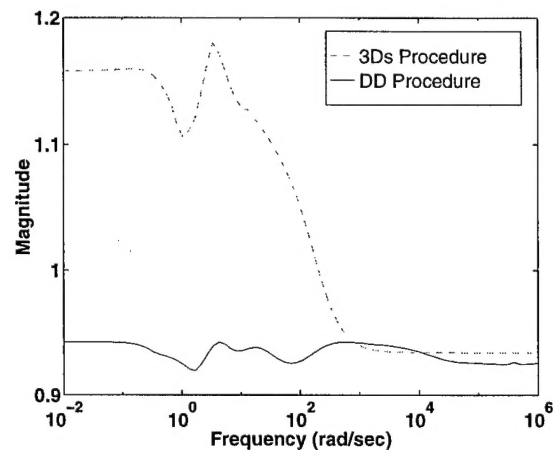

Figure 7: Example 2, Robustness Bounds using $3Ds$ and DD

5   Summary

A new technique for computing the robustness bound for systems with both TV and LTI uncertainties was presented. The method provides less conservative results than current techniques. Furthermore, it can be used on any system that

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has TV and LTI uncertainty in linear fractional form.

While the numerical results presented considered real scales, complex scales could have been used as well. However, in [6] and [10] it was shown that allowing the constant scales to be complex doesn't reduce the scaled singular value if the system being analyzed has only real state space entries. Furthermore, in [11], Packard shows that restricting scales associated with full complex LTI blocks to be positive real can be done without loss of generality. That leaves only the case of repeated LTI uncertainty with hope of benefitting from the use of complex scales. There may be cases where this improvement is significant but it hasn't been a factor in the problems we have encountered.

Since the first step of the analysis is more computationally intensive than the second it is often advantageous to obtain the suboptimal constant scales using a few strategically selected frequency points. Then, step two can be performed using a finer frequency grid to give a robustness bound. It should be noted that using this approach it is possible for the peak of the bound obtained in the second step to be larger than the peak obtained in the first step, thus indicating a frequency point that should be included in the first step.

Finally, while a set of LMIs is mathematically equivalent to a single LMI containing the set of LMIs on its diagonal, we have found that the first case requires significantly less memory and often is much faster to solve.

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A Appendix

The state space matrices A, B, C, and D of the weighted closed-loop system, G, used in example one are given below, where the closed loop matrix $F_u(G, \tilde{\Delta}) = A + B\tilde{\Delta}(I - D\tilde{\Delta})^{-1}C$.

$$A = \begin{bmatrix}
-1.06 & 0.99 & 0 & 0 & 0 & -0.17 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2.41 & -1.30 & 0 & 0 & 0 & -13.90 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.10 & 1.41 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.00 & 0 & -10.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -100.00 & -9.49 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -20.20 & -6.04 & 96.48 & -169.71 & 1435.23 & 35.68 & -28.87 \\
217.00 & 17235.05 & 0 & 0 & 0 & 0 & -22.69 & -1716.58 & -0.00 & 0.00 & 0 & -0.17 \\
17235.05 & 1394626.39 & 0 & 0 & 0 & 0 & -1719.98 & -138983.71 & -0.00 & 0.00 & 0 & -13.90 \\
0.80 & -0.01 & 0 & 0 & 0 & 0 & -0.08 & 0.00 & -0.10 & 1.41 & 0 & 0 \\
0.99 & 10.02 & 0 & 0 & 0 & 0 & -0.10 & 0.00 & -0.00 & -10.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -100.00 & -9.49 \\
0 & 0 & 0 & 0 & 0 & 0 & -60.60 & 968.18 & -1702.97 & 14401.91 & 358.08 & -309.87
\end{bmatrix}$$

$$B = \begin{bmatrix}
0.15 & 0 & -0.17 & 0 & 0 & 0 \\
0 & 0.15 & -13.90 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.02 & 1.72 \\
0 & 0 & 0 & 0 & 1.72 & 139.46 \\
0 & 0 & 0 & 0 & 0.00 & -0.00 \\
0 & 0 & 0 & -10.03 & 0.00 & 0.00 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$C = \begin{bmatrix}
-2.15 & 0.00 & 0 & 0 & 0 & -0.36 & 0 & 0 & 0 & 0 & 0 & 0 \\
-13.87 & -2.53 & 0 & 0 & 0 & -58.34 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 9.49 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.41 & -20.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.01 & 0.11 & -0.20 & 1.69 & 0.04 & -0.03 \\
0 & 0 & 0 & 0 & 0 & -0.20 & -0.06 & 0.96 & -1.70 & 14.35 & 0.36 & -0.29
\end{bmatrix}$$

$$D = \begin{bmatrix}
0 & 0 & -0.36 & 0 & 0 & 0 \\
0 & 0 & -58.34 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$